SYDE 312 NUMERICAL METHODS - FINAL EXAM

6 April 2004, 2:00-5:00

INSTRUCTIONS:

1. Aids allowed:
   - Any written material (text, notes, etc.)
   - Computer with Matlab, Excel, and other computational software available, including installed help systems
   - You may use the local C: drive to save m-files created during the exam period if you wish
   - Calculator
2. Aids NOT ALLOWED:
   - External communication (internet, email etc.)
   - Access to previously-prepared computer files (N:drive etc.)
3. Read the questions CAREFULLY.
4. Each question is allocated an EQUAL number of marks. Answer as many as you can in the time provided.
5. Provide sufficient details in your solution to demonstrate that you can apply whatever method is being used. DO NOT simply give a Matlab-derived answer, unless specifically given that option in the question.
6. Complete one blank front page with your personal information (NAME, ID).
7. Write your solutions on the answer pages provided, using both sides, and at least one complete side for each question. DO NOT MIX QUESTIONS ON ANY SIDE.
8. On each answer page side write the question number you are answering and your name in the spaces provided.
9. At the end of the exam you will assemble your pages, with the blank ID page in front on top. Have them ready to be stapled together as they are collected.

1. The following data:

x	2	3	6	8	12
y	14	20	17	16	23

determines a function $y = f(x)$.

   (a) Find a 4th degree Newton interpolating polynomial for this data.

   (b) Use Matlab (or otherwise) to find the equations for the piecewise cubic splines that interpolate the given data with knot-a-knot end conditions.

   (c) Sketch a plot of the curves you obtained in (a) and (b), clearly indicating where/how they differ.

   (d) Determine estimated values for the derivative $f'(5)$ using the two interpolating functions in parts (a) and (b). Why are the values different?

2. Solve the initial value problem

$$y' = \frac{3x}{y} - xy, y(0) = 2$$

over the interval $[0, 1]$.

   (a) Use the explicit midpoint method with step size 0.5

   (b) Use the Runge-Kutta 4th order method with step size 0.5

   (c) Use an appropriate Matlab ode solver.

   (d) Provide a sketch plot showing clearly the solution curve from part (c), and the (x, y)-values you obtained in parts (a) and (b).

3. Consider the linear system $Ax = b$ with $A = \begin{bmatrix} 4 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$ and $b = \begin{bmatrix} 3 & -1 & 4 \end{bmatrix}^T$.

   (a) Is A positive-definite?

   (b) Obtain an exact solution to the system using the quickest method (you may simply give a Matlab answer here if you wish, but explain how you obtained it).

   (c) Apply SIX iterations of the Gauss-Seidel method with a starting vector $[\ 0 \quad 0 \quad 0\ ]^T$.

   (d) Determine the 2-norm absolute error at each stage in the iteration process of part (c), comparing the current solution to the exact solution obtained in part (b).

   (e) Sketch a plot of the absolute error values you found in part (d) against the iteration number.

4. (a) Why is Gaussian-quadrature able to achieve such good accuracy with a small number of function evaluations? Are there any functions for which a Newton-Cotes rule might be better?

   (b) Provide a sketch plot of the function $f(x) = (1 + e^x)^{-2}$ over the interval $[0, 3]$.

(c) Use 3-point Gauss-Legendre quadrature to evaluate the integrals: $\int_0^3 f(x)dx$, $\int_0^{30} f(x)dx$, and $\int_0^{300} f(x)dx$. [3-point Nodes: $-\sqrt{3/5}, 0, \sqrt{3/5}$; Weights: $5/9, 8/9, 5/9$.]

(d) Evaluate the integral $\int_0^{\infty} f(x)dx$ using 3-point Gauss-Laguerre quadrature. [Nodes: 0.41577455678348, 2.29428036027904, 6.28994508293748; Weights: 0.71109300992917, 0.27851773356924, 0.01038925650159]

(e) Explain why the sequence of integrals obtained using Gauss-Legendre quadrature fails to converge to the value obtained in part (e).

5. For the function $f(x) = e^x + x^2 - 5x$:

   (a) Use a Matlab plot to estimate the location of the two roots.

   (b) Locate the smaller root to an accuracy of at least six decimals using fixed-point iteration.

   (c) Give the result of a test which predicts the failure of fixed point iteration when attempting to find the other root.

   (d) Find the other root accurate to at least six decimals using Newton's method.

   (e) Provide a sketch plot of the function, indicating the two roots you found above.

6. Consider the linear system $Ax = b$ with $A = \begin{bmatrix} 2 & 4 & 1 \\ 5 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ and $b = [\ -5 \quad 12 \quad 3\ ]^T$.

   (a) Use Matlab, or otherwise, to find an LU-factorization with partial pivoting for the matrix A, i.e. matrices L, U and a permutation matrix P, so that $PA = LU$.

   (b) Use the factorization of part (a) to obtain (by hand) the exact solution to the system.

   (c) The vector $\hat{x} = [\ 2 \quad -5 \quad 12\ ]^T$ is an approximate solution to the system. Apply iterative improvement to this $\hat{x}$ and attempt to obtain the solution you found in part (b).

7. The following data: determines y as a function of x.

x	-1	0	1	3
y	13.65	1.38	0.49	0.15

   (a) Linearize the data with the transformation $Y = \ln(y)$ and find a least-squares fit of the form $f(x) = ce^{ax}$. Give the normal equations and solve them using a QR factorization (which you can obtain with Matlab if you wish).

   (b) Linearize the data with the transformation $Y = y^{-1/2}$ and find a least-squares fit of the form $f(x) = (ax + b)^{-2}$. Give the normal equations and solve them using the SVD (which you can obtain with Matlab if you wish).

   (c) Determine which of the fitting functions above is better, by evaluating the residuals with respect to the original data.